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Output Feedback Design in Discrete-Time Control Systems as an Optimization Problem

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Abstract—This paper proposes a novel approach to rejecting nonrandom bounded exogenous disturbances in linear discrete-time control systems by means of dynamic output feedback. The approach is based on reducing the original problem to a matrix optimization problem with the gain and observer matrices as the variables. A gradient descent method for finding dynamic output feedback is derived and justified.

Keywords: linear systems, discrete time, exogenous disturbances, output feedback, observer, optimization, Lyapunov equation, gradient descent method, Newton's method

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1. INTRODUCTION

A new approach to rejecting nonrandom bounded exogenous disturbances in linear continuous-time control systems by means of dynamic output feedback using the Luenberger observer [2, 3] was proposed and discussed in [1]. Dating back to the work [4], this approach is based on reducing the original problem to a matrix optimization problem, where the variables are the gain and observer matrices. Below, this problem is solved using a gradient descent method.

The literature on the corresponding range of problems was briefly reviewed in [1]; for example, see [5] and the extensive bibliography therein. The same problem was solved in [6] via reduction to a semidefinite programming problem in terms of linear matrix inequalities (LMIs) [7, 8]; in doing so, several rough transformations were undertaken to linearize the matrix inequalities and establish the final result in terms of LMIs. Consequently, the final conditions suffered from excessive conservatism.

Among ideologically similar publications, note the recent works [9, 10], where a promising approach was proposed based on solving parameterized matrix Riccati equations.

This paper is a direct continuation of the research [1], extending the considerations presented therein to discrete-time control systems. An important applications-oriented feature of the approach is the possibility of limiting the magnitude of the observer and dynamic controller matrices.

From now on, $\mathbb{S}^n$ denotes the space of symmetric real matrices of dimensions $n \times n$; $\|\cdot\|$ is the Euclidean norm of a vector and the spectral norm of a matrix; $\|\cdot\|_F$ indicates the Frobenius norm of a matrix; $^{\mathrm{T}}$ is the transpose symbol; tr stands for the trace of a matrix; $\langle \cdot, \cdot \rangle$ means the Frobenius inner product of matrices; I denotes an identity matrix of appropriate dimension; $\lambda_i(A)$ are the eigenvalues of a matrix A; finally, $\rho(A) = \max_i |\lambda_i(A)| < 1$ specifies the stability radius of a Schur matrix A.

2. PROBLEM STATEMENT

Consider a control system described by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Dw_k, \\ y_k &= C_1x_k + D_1w_k, \\ z_k &= C_2x_k, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{n\times m}$, $D_1 \in \mathbb{R}^{\ell\times m}$, $C_1 \in \mathbb{R}^{\ell\times n}$, $C_2 \in \mathbb{R}^{r\times n}$, with the state $x_k \in \mathbb{R}^n$, an initial condition x_0, the observed output $y_k \in \mathbb{R}^\ell$, the optimized (target) output $z_k \in \mathbb{R}^r$, the control input $u_k \in \mathbb{R}^p$, and an exogenous disturbance[1] $w_k \in \mathbb{R}^m$ bounded at each time instant:

$$\|w_k\| \leqslant \delta \quad \text{for all } k = 0, 1, 2, \ldots.$$

By assumption, the pairs (A, B) and (A, C_1) are controllable and observable, respectively.

Suppose that the state vector x_k of the system is unmeasurable and information about the system is provided by its output y_k. The problem is to find a minimal (in a sense) ellipsoid containing the target output z_k.

Let us construct an observer described by a linear difference equation with the mismatch between the output y_k and its prediction $C_1\widehat{x}_k$:

$$\widehat{x}_{k+1} = A\widehat{x}_k + Bu_k + L(y_k - C_1\widehat{x}_k), \quad \widehat{x}_0 = 0, \tag{2}$$

where $L \in \mathbb{R}^{n\times\ell}$ is the observer matrix.

According to (1) and (2), the error $e_k = x_k - \widehat{x}_k$ satisfies the difference equation

$$e_{k+1} = (A - LC_1)e_k + (D - LD_1)w_k, \quad e_0 = x_0.$$

With the dynamic feedback controller

$$u_k = K\widehat{x}_k, \quad K \in \mathbb{R}^{p\times n}, \tag{3}$$

applied to system (1), we arrive at the closed loop system

$$\begin{aligned} x_{k+1} &= (A + BK)x_k - BKe_k + Dw_k, \\ e_{k+1} &= (A - LC_1)e_k + (D - LD_1)w_k, \\ z_k &= C_2x_k \end{aligned} \tag{4}$$

with the controlled output z_k.

Like in the continuous-time case, it is often impossible to design a static output-feedback controller $u_k = Ky_k$ for the system: the matrix $A + BKC_1$ may turn out nonstabilizable by the choice of K, while the dynamic controller (3) can be constructed (under the relatively mild requirements of system controllability and observability; see Section 3 for details).

3. THE SOLUTION APPROACH

We employ the invariant ellipsoid method [8, 13]. Recall that an ellipsoid

$$\mathcal{E}_x = \{x \in \mathbb{R}^n\colon \quad x^{\mathrm{T}}P^{-1}x \leqslant 1\}, \quad P \succ 0,$$

[1] Although the nature of disturbances in the state vector and output of the system generally differs, it is convenient to consider them the same, assuming that the matrices D and D_1 "cut out" different "pieces" from the vector w_k; the general case can also be considered at the expense of some complication.

is said to be *invariant* for a linear discrete-time dynamic system

$$\begin{aligned} x_{k+1} &= Ax_k + Dw_k, \quad \|w_k\| \leqslant 1, \\ z_k &= Cx_k \end{aligned} \tag{5}$$

with a Schur matrix A if any system trajectory starting from an initial point x_0 inside the ellipsoid $\mathcal{E}_x$ remains there at any time instant under all exogenous disturbances $w_k\colon \|w_k\| \leqslant 1$.

An invariant ellipsoid has the attraction property: a system trajectory starting from an initial point outside the ellipsoid tends to it over time.

Obviously, if $\mathcal{E}_x$ is an invariant ellipsoid with a matrix P, the linear output z_k of system (5) with $x_0 \in \mathcal{E}_x$ will belong to the ellipsoid

$$\mathcal{E}_z = \{z \in \mathbb{R}^r\colon \quad z^{\mathrm{T}}(CPC^{\mathrm{T}})^{-1}z \leqslant 1\},$$

called *bounding*. (In the case $x_0 \notin \mathcal{E}_x$, this output will tend to the bounding ellipsoid.)

When estimating the effect of exogenous disturbances on the system output, minimal bounding ellipsoids are of natural interest; a common minimality criterion for a bounding ellipsoid is the value $\operatorname{tr} CPC^{\mathrm{T}}$, equal to the sum of the squares of its semiaxes. The following result is well known; for example, see the book [8].

Theorem 1. *Assume that A is a Schur matrix, $\rho = \max_i |\lambda_i(A)| < 1$, the pair (A, D) is controllable, and a matrix $P(\alpha) \succ 0$, $\rho^2 < \alpha < 1$, satisfies the discrete Lyapunov equation*

$$\frac{1}{\alpha}APA^{\mathrm{T}} - P + \frac{1}{1-\alpha}DD^{\mathrm{T}} = 0. \tag{6}$$

Then the problem on the optimal bounding ellipsoid for system (5) *is reduced to minimizing the one-dimensional objective function*

$$f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}}$$

on the interval $\rho^2 < \alpha < 1$.

Let us make several remarks. First, if α^* is a minimum point in the problem of Theorem 1 and x_0 satisfies the condition $x_0^{\mathrm{T}}P^{-1}(\alpha^*)x_0 \leqslant 1$, then the following bound uniformly holds:

$$\|z_k\|^2 \leqslant \|CP(\alpha^*)C^{\mathrm{T}}\| \leqslant \operatorname{tr} CP(\alpha^*)C^{\mathrm{T}} = f(\alpha^*), \quad k = 0, 1, 2, \ldots.$$

Second, equation (6) can be represented as

$$\left(\frac{1}{\sqrt{\alpha}}A\right) P \left(\frac{1}{\sqrt{\alpha}}A\right)^{\mathrm{T}} - P + \frac{1}{1-\alpha}DD^{\mathrm{T}} = 0;$$

by [8, Lemma 1.2.6], it has a unique positive definite solution if and only if $\frac{1}{\sqrt{\alpha}}A$ is a Schur matrix, i.e.,

$$\rho\left(\frac{1}{\sqrt{\alpha}}A\right) < 1.$$

Introducing the combined vector

$$g_k = \begin{pmatrix} x_k \\ e_k \end{pmatrix} \in \mathbb{R}^{2n},$$

we write system (4) in matrix form and consider the matrices $A_{K,L}$, D_L, and $\mathcal{C}$:

$$\dot{g}_k = \underbrace{\begin{pmatrix} A+BK & -BK \\ 0 & A-LC_1 \end{pmatrix}}_{A_{K,L}} g_k + \underbrace{\begin{pmatrix} D \\ D-LD_1 \end{pmatrix}}_{D_L} w_k, \quad g_0 = \begin{pmatrix} x_0 \\ x_0 \end{pmatrix},$$
$$z_k = \underbrace{\begin{pmatrix} C_2 & 0 \end{pmatrix}}_{\mathcal{C}} g_k. \tag{7}$$

Enclosing the state g_k of system (7) in the invariant ellipsoid

$$\mathcal{E}_g = \{g \in \mathbb{R}^{2n}: \quad g^{\mathrm{T}} P^{-1} g \leqslant 1\}$$

generated by a matrix $P \in \mathbb{S}^{2n}$, we minimize the size of the corresponding bounding ellipsoid

$$\mathcal{E}_z = \{z \in \mathbb{R}^{r}: \quad z^{\mathrm{T}} (\mathcal{C} P \mathcal{C}^{\mathrm{T}})^{-1} z \leqslant 1\}$$

for the output z_k with the matrix $\mathcal{C}P\mathcal{C}^{\mathrm{T}}$. As the minimality criterion we choose the trace function $\operatorname{tr} \mathcal{C}P\mathcal{C}^{\mathrm{T}}$.

Note that the matrix $A_{K,L}$ can be represented as

$$A_{K,L} = \begin{pmatrix} A+BK & -BK \\ 0 & A-LC_1 \end{pmatrix} = \underbrace{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}}_{\mathcal{A}} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{M_1} K \underbrace{\begin{pmatrix} I & -I \end{pmatrix}}_{N_1} + \underbrace{\begin{pmatrix} 0 \\ I \end{pmatrix}}_{M_2} L \underbrace{\begin{pmatrix} 0 & -C_1 \end{pmatrix}}_{N_2}.$$

According to Theorem 1, we therefore arrive at the problem of minimizing the objective function $\operatorname{tr} \mathcal{C}P\mathcal{C}^{\mathrm{T}}$ subject to the constraint

$$\frac{1}{\alpha}(\mathcal{A} + M_1 K N_1 + M_2 L N_2) P (\mathcal{A} + M_1 K N_1 + M_2 L N_2)^{\mathrm{T}} - P + \frac{\delta^2}{1-\alpha} D_L D_L^{\mathrm{T}} = 0 \tag{8}$$

with respect to the matrix variables $0 \prec P \in \mathbb{S}^{2n}$, $K \in \mathbb{R}^{p\times n}$, and $L \in \mathbb{R}^{n\times \ell}$ and the scalar parameter $\alpha > 0$.

As the performance criterion we take the function

$$f(K, L, \alpha) = \operatorname{tr} \mathcal{C}P\mathcal{C}^{\mathrm{T}} + \chi_K \|K\|_F^2 + \chi_L \|L\|_F^2, \tag{9}$$

which incorporates, in addition to the component determining the size of the bounding ellipsoid by the trace criterion, penalties for the magnitude of the gain and observer matrices (the coefficients χ_K and $\chi_L > 0$ tune their significance). Their presence ensures the coercivity of the objective function in K and L; for details, see Section 4.

Remark 1. Note that the block matrix $A_{K,L}$ has the same eigenvalues as the matrices $A + BK$ and $A - LC_1$ standing on its diagonal. In turn, the existence of matrices K and L such that the matrices $A + BK$ and $A - LC_1$ are stable follows from the controllability and observability of the original system.

By Remark 1, there surely exist matrices K_0 and L_0 such that A_{K_0,L_0} is a Schur matrix. Matrices (K, L) with this property will be called a *stabilizing matrix pair*.

4. OPTIMIZATION OF THE FUNCTION $f(K, L, \alpha)$

Thus, the original problem of designing an observer-aided dynamic feedback controller rejecting the effect of exogenous disturbances on the output z_k of system (1) has been reduced to that of minimizing the objective function $f(K, L, \alpha)$ (9) subject to the constraint

$$\frac{1}{\alpha} A_{K,L} P A_{K,L}^{\mathrm{T}} - P + \frac{\delta^2}{1-\alpha} D_L D_L^{\mathrm{T}} = 0 \tag{10}$$

with respect to the matrix variables $P \in \mathbb{S}^{2n}$, $K \in \mathbb{R}^{p\times n}$, and $L \in \mathbb{R}^{n\times \ell}$ and the scalar parameter $\alpha > 0$. The notation $f(K, L, \alpha)$ emphasizes that given K, L, and α, the matrix P is found from the Lyapunov equation (10); therefore, K, L, and α are the independent variables.

The properties of the function $f(\alpha) = \operatorname{tr} \mathcal{C} P \mathcal{C}^{\mathrm{T}}$ established in [11] (under some fixed stabilizing pair (K, L)) can be fully transferred to the case under consideration. In particular, the function $f(\alpha)$ is well-defined, positive, and strictly convex on the interval $\rho^2(A_{K,L}) < \alpha < 1$, and its values tend to infinity at the ends of this interval.

The function $f(\alpha)$ can be effectively minimized using Newton's method. Let us specify some initial approximation $\rho^2(A_{K,L}) < \alpha_0 < 1$, e.g., $\alpha_0 = \left(1 + \rho^2(A_{K,L})\right)/2$, and apply the iterative process

$$\alpha_{j+1} = \alpha_j - \frac{f'(\alpha_j)}{f''(\alpha_j)}. \tag{11}$$

Here, according to [11],

$$f'(\alpha) = \operatorname{tr} Y \left(\frac{\delta^2}{(1-\alpha)^2} D_L D_L^{\mathrm{T}} - \frac{1}{\alpha^2} A_{K,L} P A_{K,L}^{\mathrm{T}} \right),$$

$$f''(\alpha) = 2 \operatorname{tr} Y \left(\frac{\delta^2}{(1-\alpha)^3} D_L D_L^{\mathrm{T}} + \frac{1}{\alpha^3} A_{K,L} (P - X) A_{K,L}^{\mathrm{T}} \right),$$

with P, Y, and X representing the solutions of the discrete Lyapunov equations (10),

$$\frac{1}{\alpha} A_{K,L}^{\mathrm{T}} Y A_{K,L} - Y + \mathcal{C}^{\mathrm{T}} \mathcal{C} = 0,$$

and

$$\frac{1}{\alpha} A_{K,L} X A_{K,L}^{\mathrm{T}} - X + \frac{\delta^2}{(1-\alpha)^2} D_L D_L^{\mathrm{T}} - \frac{1}{\alpha^2} A_{K,L} P A_{K,L}^{\mathrm{T}} = 0,$$

respectively.

The next result ensures the global convergence of the algorithm.

Theorem 2 [11]. *In the method* (11), *we have the upper bounds*

$$|\alpha_j - \alpha^*| \leqslant \frac{f''(\alpha_0)}{2^j f''(\alpha^*)} |\alpha_0 - \alpha^*|, \qquad |\alpha_{j+1} - \alpha^*| \leqslant c |\alpha_j - \alpha^*|^2,$$

where $c > 0$ is some constant (*which can be written explicitly*).

Due to the first bound, the method converges globally (faster than a geometric progression with a ratio of 1/2); due to the second bound, it converges quadratically in the vicinity of the solution.

We pass to the minimization of the function

$$f(K, L) \doteq \min_{\alpha} f(K, L, \alpha),$$

after examining its properties.

Lemma 1 [1]. *The function $f(K,L)$ is well-defined and positive on the set $\mathcal{S}$ of all stabilizing matrix pairs.*

The definitional domain $\mathcal{S}$ of the function $f(K,L)$ may be nonconvex and disconnected and, as in the continuous-time case, its boundaries may be nonsmooth (see [12]).

Lemma 2. *The function $f(K,L,\alpha)$ is well-defined for $(K,L)\in\mathcal{S}$ and $\rho^2(A_{K,L})<\alpha<1$. On this admissible set, it is differentiable and the gradient is given by*

$$\nabla_\alpha f(K,L,\alpha) = \operatorname{tr} Y\left(\frac{\delta^2}{(1-\alpha)^2}D_L D_L^{\mathrm{T}} - \frac{1}{\alpha^2}A_{K,L}PA_{K,L}^{\mathrm{T}}\right),$$

$$\frac{1}{2}\nabla_K f(K,L,\alpha) = \frac{1}{\alpha}M_1^{\mathrm{T}}YA_{K,L}PN_1^{\mathrm{T}} + \chi_K K, \tag{12}$$

$$\frac{1}{2}\nabla_L f(K,L,\alpha) = \frac{1}{\alpha}M_2^{\mathrm{T}}YA_{K,L}PN_2^{\mathrm{T}} - \frac{\delta^2}{1-\alpha}\begin{pmatrix}0 & I\end{pmatrix}YD_L D_1^{\mathrm{T}} + \chi_L L, \tag{13}$$

where the matrix Y satisfies the discrete Lyapunov equation

$$\frac{1}{\alpha}A_{K,L}^{\mathrm{T}}YA_{K,L} - Y + \mathcal{C}^{\mathrm{T}}\mathcal{C} = 0. \tag{14}$$

The function $f(K,L,\alpha)$ achieves minimum at an interior point of the admissible set under the necessary conditions

$$\nabla_K f(K,L,\alpha)=0, \quad \nabla_L f(K,L,\alpha)=0, \quad \nabla_\alpha f(K,L,\alpha)=0.$$

In addition, $f(K,L,\alpha)$ as a function of α is strictly convex on $\rho^2(A_{K,L})<\alpha<1$ and achieves minimum at an interior point of this interval.

The proofs of this and subsequent statements are provided in the Appendix.

Next, to obtain simple quantitative estimates in Lemma 3, we embed a regularizing additive term ε into the objective function (9) as follows:

$$f(K,L,\alpha) = \operatorname{tr} P(\mathcal{C}^{\mathrm{T}}\mathcal{C}+\varepsilon I) + \chi_K\|K\|_F^2 + \chi_L\|L\|_F^2 \to \min_{K,L,\alpha}, \quad 0<\varepsilon\ll 1.$$

The requirement for adding this term can be weakened considerably, but the current aim is to obtain the simplest and most straightforward results.

Lemma 3. *The function $f(K,L)$ is coercive on the set $\mathcal{S}$(i.e., tends to infinity on its boundary), and the following lower bounds hold:*

$$f(K,L) \geqslant \frac{\delta^2}{1-\rho^2(A_{K,L})}\frac{\varepsilon}{1-\sigma_{\min}^2(A_{K,L})}\|D_L\|_F^2, \tag{15}$$

$$f(K,L) \geqslant \chi_K\|K\|^2,$$

$$f(K,L) \geqslant \chi_L\|L\|^2.$$

We introduce the level set

$$\mathcal{S}_0 = \{(K,L)\in\mathcal{S}\colon \quad f(K,L)\leqslant f(K_0,L_0)\}.$$

Lemma 3 implies the following obvious result.

Corollary 1. *For any $(K_0,L_0)\in\mathcal{S}$ the set $\mathcal{S}_0$ is bounded.*

On the other hand, the function $f(K, L)$ achieves minimum on the set $\mathcal{S}_0$ (as a continuous function on a compact set, due to the properties of the solution of the Lyapunov equation), but the set $\mathcal{S}_0$ has no common points with the boundary of $\mathcal{S}$ by (15). According to the considerations above, $f(K, L)$ is differentiable on $\mathcal{S}_0$. Hence, we arrive at another fact.

Corollary 2. *There exists a minimum point (K_*, L_*) on the set S and the gradient of the function $f(K, L)$ vanishes at this point.*

The gradients of the function $f(K, L)$ with respect to K and L are not Lipschitz functions on the set S of all stabilizing controllers; however, it can be shown that they have this property on its subset S_0.

The obtained properties of the objective function and its derivatives allow us to build a minimization method and justify its convergence.

5. THE SOLUTION ALGORITHM

We propose the following iterative approach to solving problem (8)–(9). It is based on the alternate application of the gradient descent method with respect to the variables K and L and minimization with respect to the parameter α by Newton's method.

Algorithm for minimizing $f(K, L, \alpha)$:

Step 1. Specify values of the parameters $\varepsilon > 0$ and $0 < \tau_K, \tau_L < 1$ and an initial stabilizing matrix pair (K_0, L_0).
Calculate

$$\alpha_0 = \frac{1 + \rho^2(\mathcal{A} + M_1 K_0 N_1 + M_2 L_0 N_2)}{2}.$$

Step 2. At the jth iteration, we have K_j, L_j, and α_j known.
Calculate the gradient $H_j^K = \nabla_K f(K_j, L_j, \alpha_j)$.

Step 3. Make a step of the gradient descent method in K:

$$K_{j+1} = K_j - \gamma_j^K H_j^K,$$

with a step length $\gamma_j^K > 0$ selected by splitting γ_K until the following conditions hold:
a. K_{j+1} stabilizes the matrix $(\mathcal{A} + M_1 K N_1 + M_2 L_j N_2)/\sqrt{\alpha_j}$ (i.e., turns it into a Schur matrix);
b. $f(K_{j+1}) \leqslant f(K_j) - \tau_K \gamma_j^K \|H_j^K\|^2$.

Step 4. Using K_{j+1}, calculate the gradient $H_j^L = \nabla_L f(K_{j+1}, L_j, \alpha_j)$.

Step 5. Make a step of the gradient descent method in L:

$$L_{j+1} = L_j - \gamma_j^L H_j^L,$$

with a step length $\gamma_j^L > 0$ selected by splitting γ_L until the following conditions hold:
a. L_{j+1} stabilizes the matrix $(\mathcal{A} + M_1 K_{j+1} N_1 + M_2 L N_2)/\sqrt{\alpha_j}$;
b. $f(L_{j+1}) \leqslant f(L_j) - \tau_L \gamma_j^L \|H_j^L\|^2$.

Step 6. For the resulting matrices K_{j+1} and L_{j+1}, minimize $f(K_{j+1}, L_{j+1}, \alpha)$ with respect to α to obtain α_{j+1}. Go back to Step 2.
The *normalization condition* is given by

$$\|H_j^K\| \leqslant \varepsilon, \quad \|H_j^L\| \leqslant \varepsilon.$$

In this case, take the current pair (K_j, L_j) as an approximate solution and stop the algorithm.

An important point is the choice of the trial step of the gradient descent method. Here, it seems promising to use the following considerations. For some K, L, α, and $P \succ 0$, let

$$\frac{1}{\alpha}(\mathcal{A} + M_1KN_1 + M_2LN_2)P(\mathcal{A} + M_1KN_1 + M_2LN_2)^{\mathrm{T}} - P + \frac{\delta^2}{1-\alpha}D_LD_L^{\mathrm{T}} = 0.$$

Consider the increment with respect to K:

$$K \to K - \gamma H^K, \quad H^K = \nabla_K f(K, L, \alpha),$$

and find for which γ the matrix $\mathcal{A} + M_1(K - \gamma H^K)N_1 + M_2LN_2$ will remain stable (Schur).

To this end, it suffices to require that P remain the matrix of a quadratic Lyapunov function for $\mathcal{A} + M_1(K - \gamma H^K)N_1 + M_2LN_2$, i.e.,

$$(\mathcal{A} + M_1(K - \gamma H^K)N_1 + M_2LN_2)P(\mathcal{A} + M_1(K - \gamma H^K)N_1 + M_2LN_2)^{\mathrm{T}} - P \prec 0,$$

or

$$\begin{pmatrix} P & \mathcal{A} + M_1(K - \gamma H^K)N_1 + M_2LN_2 \\ (\mathcal{A} + M_1(K - \gamma H^K)N_1 + M_2LN_2)^{\mathrm{T}} & P^{-1} \end{pmatrix} \succ 0.$$

The last relation can be written as

$$\begin{pmatrix} P & A_{K,L} \\ A_{K,L}^{\mathrm{T}} & P^{-1} \end{pmatrix} - \gamma \begin{pmatrix} 0 & M_1H^KN_1 \\ (M_1H^KN_1)^{\mathrm{T}} & 0 \end{pmatrix} \succ 0,$$

which yields, according to [14],

$$0 < \gamma^K < \min_{\lambda_i > 0} \lambda_i \left(\begin{pmatrix} P & A_{K,L} \\ A_{K,L}^{\mathrm{T}} & P^{-1} \end{pmatrix}, \begin{pmatrix} 0 & M_1H^KN_1 \\ (M_1H^KN_1)^{\mathrm{T}} & 0 \end{pmatrix} \right).$$

When optimizing the objective function with respect to the variable L, the trial step is selected by analogy:

$$0 < \gamma^L < \min_{\lambda_i > 0} \lambda_i \left(\begin{pmatrix} P & A_{K,L} \\ A_{K,L}^{\mathrm{T}} & P^{-1} \end{pmatrix}, \begin{pmatrix} 0 & M_2H^LN_2 \\ (M_2H^LN_2)^{\mathrm{T}} & 0 \end{pmatrix} \right),$$

where $H^L = \nabla_L f(K, L, \alpha)$.

6. AN ILLUSTRATIVE EXAMPLE: CONTROL OF A TWO-MASS SYSTEM

Consider a *two-mass system* consisting of two rigid bodies with masses m_1 and m_2 connected by a spring with an elasticity coefficient k. Assume that the system slides without friction along a fixed horizontal rod (see Fig. 1). A control action u is applied to the left body, and each of the bodies is subjected to a corresponding exogenous disturbance (w_1 or w_2).

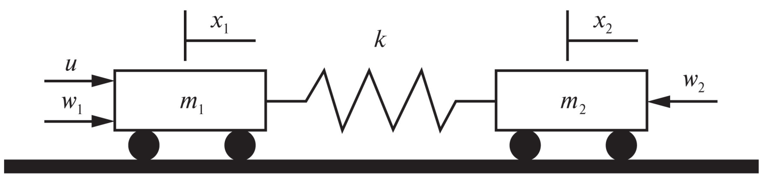

Fig. 1. A two-mass system.

Let x_1, v_1 and x_2, v_2 denote the coordinate and velocity of the left and right bodies, respectively. The observed output is the noisy two-dimensional vector

$$y = \begin{pmatrix} x_1 \\ x_2 + w_3 \end{pmatrix},$$

and the controlled (target) output is

$$z = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Assume that the combined vector of exogenous disturbances is bounded at each time instant:

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad \|w\| \leqslant 0.2.$$

For the unit parameters of the two-mass system and the sampling interval $\Delta = 0.1$, we arrive at system (1) with the matrices

$$A = \begin{pmatrix} 0.9950 & 0.0050 & 0.0998 & 0.0002 \\ 0.0050 & 0.9950 & 0.0002 & 0.0998 \\ -0.0997 & 0.0997 & 0.9950 & 0.0050 \\ 0.0997 & -0.0997 & 0.0050 & 0.9950 \end{pmatrix}, \quad B = \begin{pmatrix} 0.0050 \\ 0.0000 \\ 0.0998 \\ 0.0002 \end{pmatrix}, \quad D = \begin{pmatrix} 0.0050 & 0 & 0 \\ 0 & 0.0050 & 0 \\ 0.0998 & 0.0002 & 0 \\ 0.0002 & 0.0998 & 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We specify an initial stabilizing pair of the form

$$K_0 = \begin{pmatrix} -6.9566 & 2.6036 & -4.1281 & -5.8682 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 3.2037 & -0.6077 \\ 3.5790 & 0.3961 \\ 3.3299 & 0.1006 \\ -7.0446 & 5.3352 \end{pmatrix}$$

and let

$$\chi_K = \chi_L = 0.1.$$

In accordance with the algorithm above, the iterative procedure gives the gain matrix

$$K^* = \begin{pmatrix} -1.0311 & 0.6922 & -1.8064 & 0.0088 \end{pmatrix}, \quad \|K^*\| = 2.1921,$$

of the dynamic controller, the observer matrix

$$L^* = \begin{pmatrix} 1.1630 & -0.1759 \\ 1.8237 & 0.3432 \\ 2.0765 & 0.0714 \\ -3.7874 & 2.6783 \end{pmatrix}, \quad \|L^*\| = 5.2735,$$

and the corresponding bounding ellipse with the matrix

$$\mathcal{C}P^*\mathcal{C}^{\mathrm{T}} = \begin{pmatrix} 0.3974 & 0.1478 \\ 0.1478 & 0.4115 \end{pmatrix}, \quad \operatorname{tr} \mathcal{C}P^*\mathcal{C}^{\mathrm{T}} = 0.8089;$$

in this case, $\alpha^* = 0.9714$. The dynamics of the values of the objective function $f(K, L)$ are shown in Fig. 2.

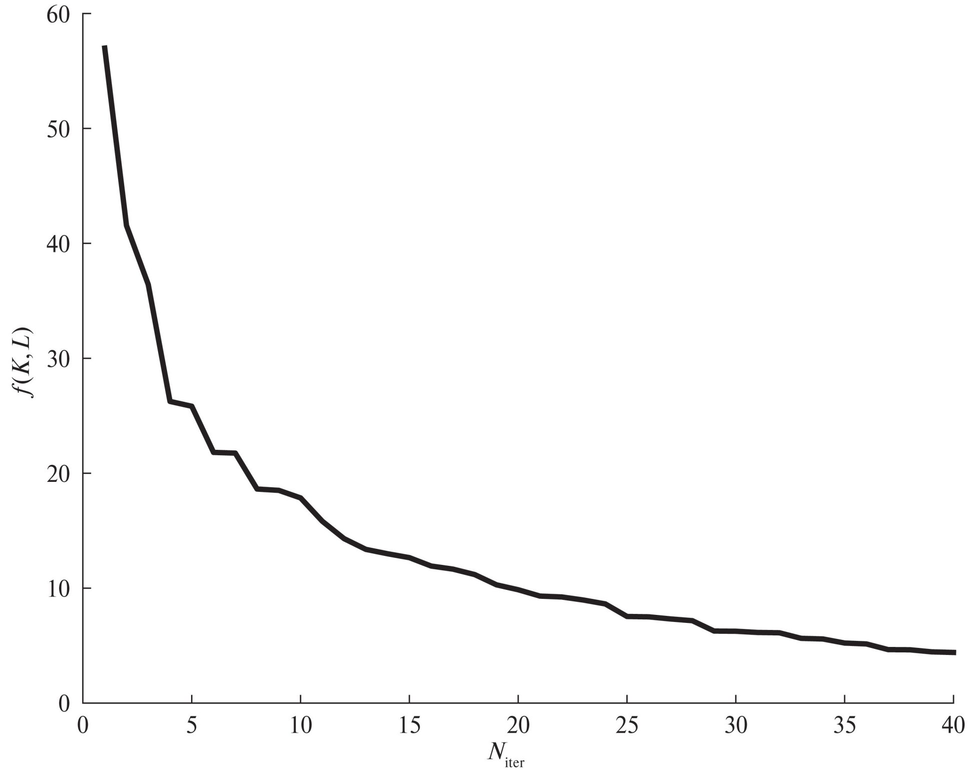

Fig. 2. The optimization procedure under $\chi_K = \chi_L = 0.1$.

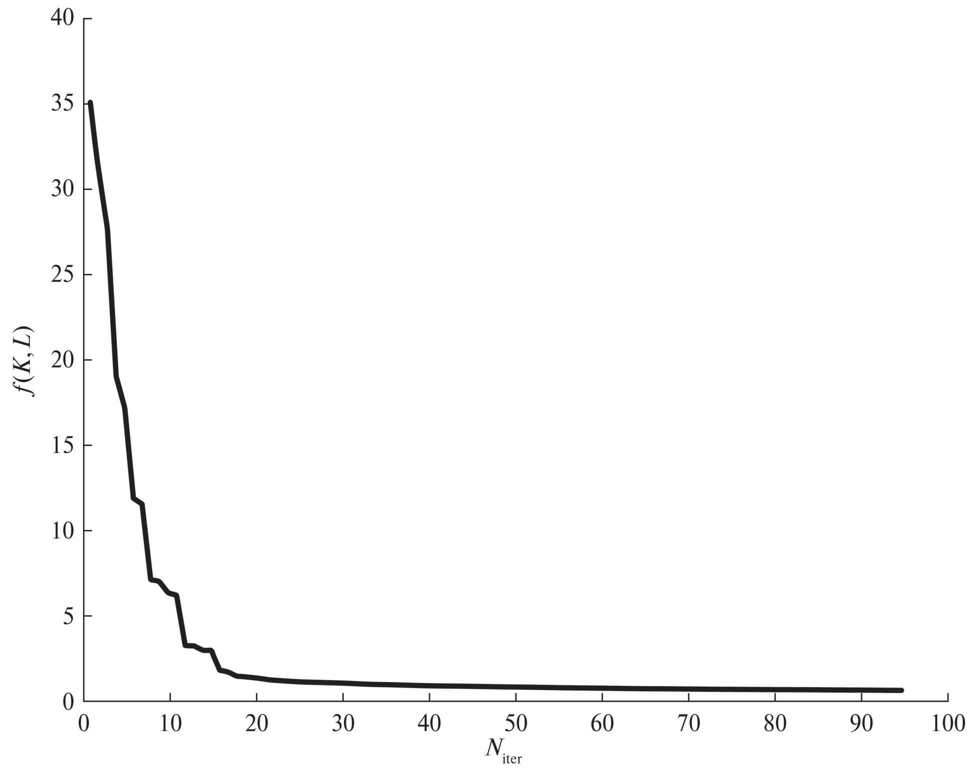

Fig. 3. The optimization procedure under $\chi_K = \chi_L = 0.01$.

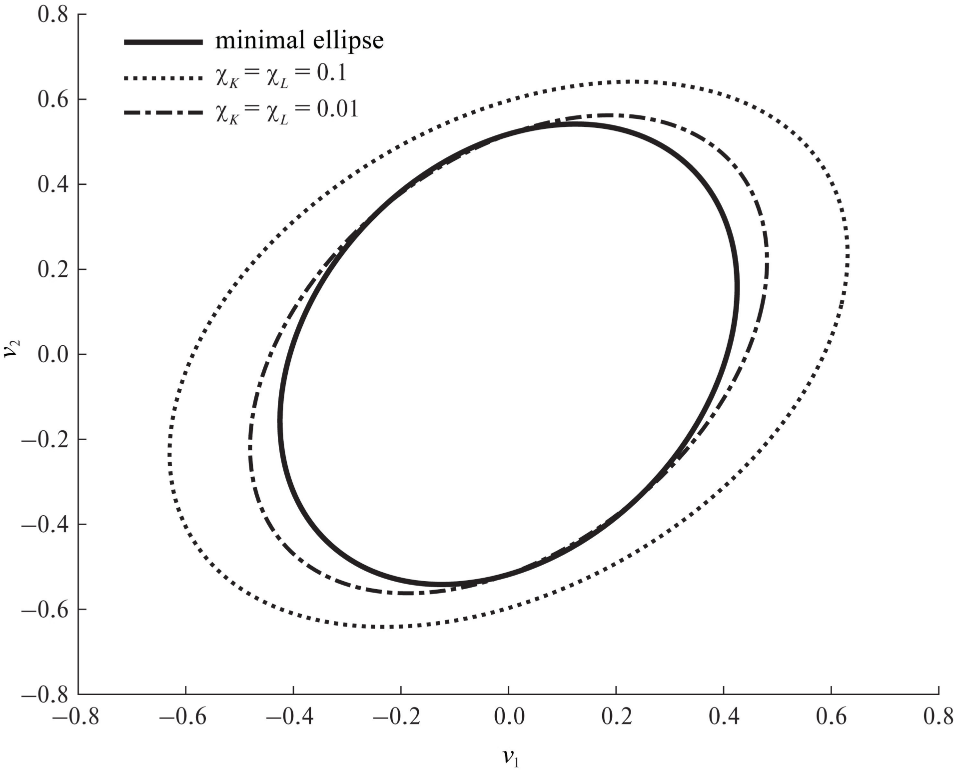

Fig. 4. Bounding ellipses.

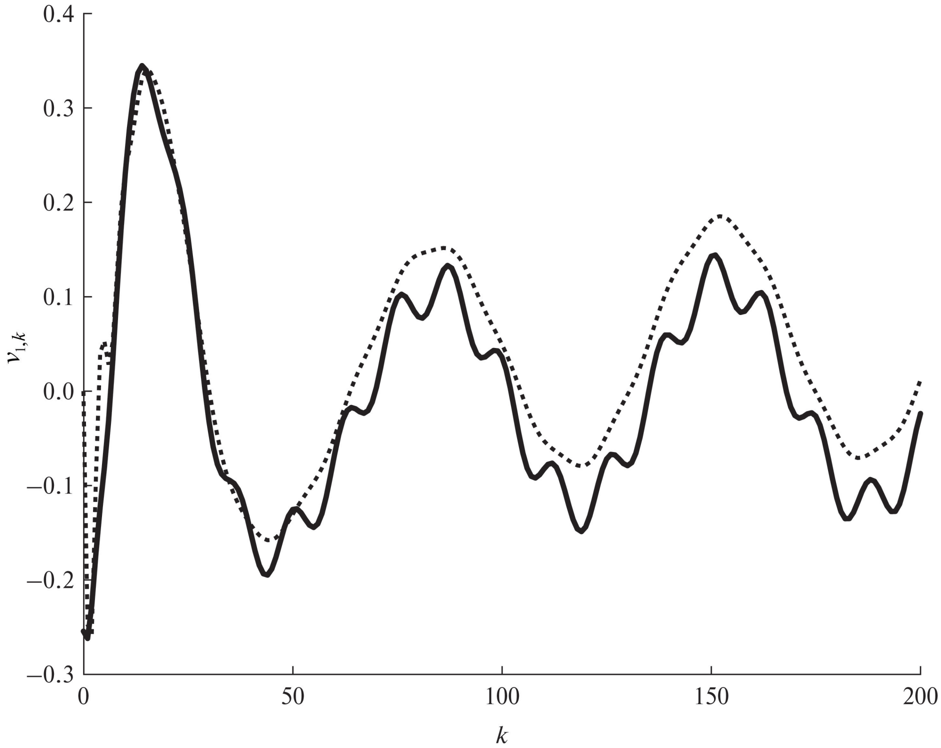

Fig. 5. The dynamics of the coordinate v_1 (solid line) and its estimate $\widehat{v}_1$ (dotted line).

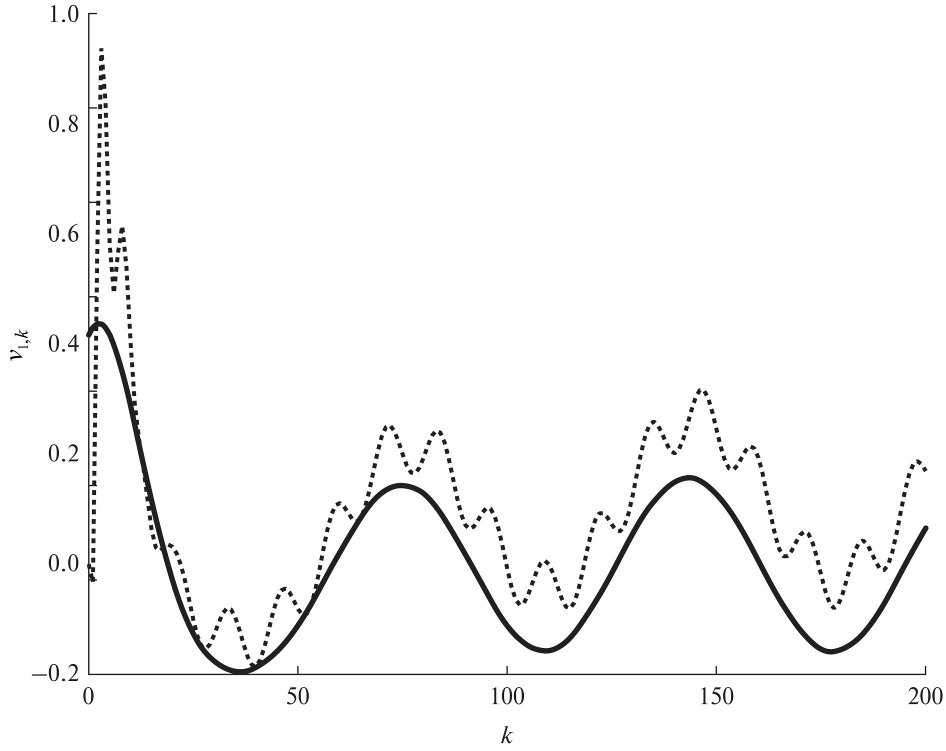

Fig. 6. The dynamics of the coordinate v_2 (solid line) and its estimate $\widehat{v}_2$ (dotted line).

Now, assigning

$$\chi_K = \chi_L = 0.01,$$

we obtain the gain matrix

$$K^{'*} = \begin{pmatrix} -1.3591 & 0.8130 & -2.5922 & 0.0186 \end{pmatrix}, \quad \|K^{'*}\| = 3.0378,$$

the observer matrix

$$L^{'*} = \begin{pmatrix} 0.6899 & -0.0066 \\ 2.7589 & 0.5861 \\ 3.0458 & 0.1396 \\ -5.3792 & 3.8014 \end{pmatrix}, \quad \|L^{'*}\| = 7.4014$$

and the corresponding bounding ellipse with the matrix

$$\mathcal{C}P^{'*}\mathcal{C}^{\mathrm{T}} = \begin{pmatrix} 0.2310 & 0.1056 \\ 0.1056 & 0.3163 \end{pmatrix}, \quad \operatorname{tr}\mathcal{C}P^{'*}\mathcal{C}^{\mathrm{T}} = 0.5473;$$

in this case, $\alpha^* = 0.9677$. The dynamics of the values of the objective function $f(K, L)$ are shown in Fig. 3.

Obviously, the reduction of the penalty coefficients produced an expected outcome: the size of the bounding ellipse decreased (by about one-third), at the cost of increasing the norm of the gain and observer matrices (by approximately 40%).

For comparison, we solved the same problem using the approach proposed in [10]; the calculations yielded the gain matrix

$$\widehat{K} = \begin{pmatrix} -15.2436 & 10.0353 & -10.7619 & -8.9984 \end{pmatrix}, \quad \|\widehat{K}\| = 23.0188,$$

the observer matrix

$$\widehat{L} = \begin{pmatrix} 3.0329 & 0.0004 \\ 14.9866 & 0.0818 \\ 20.9524 & 0.0080 \\ 6.6502 & 0.0405 \end{pmatrix}, \quad \|\widehat{L}\| = 26.7774,$$

and the corresponding minimal bounding ellipse with the matrix

$$\widehat{R} = \begin{pmatrix} 0.1807 & 0.0672 \\ 0.0672 & 0.2935 \end{pmatrix}, \quad \operatorname{tr} \widehat{R} = 0.4742.$$

Clearly, the bounding ellipse with the matrix $\mathcal{C}P'^{*}\mathcal{C}^{\mathrm{T}}$ exceeds the optimal one (with the matrix $\widehat{R}$) by merely 15%, while the corresponding matrices of the dynamic controller and observer differ in norm by several times (almost by an order of magnitude for the gain matrix).

The solid line in Fig. 4 presents the optimal bounding ellipse obtained by the method [10]; the dotted and dash-and-dot lines show the bounding ellipses found using the optimization procedure under $\chi_K = \chi_L = 0.1$ and $\chi_K = \chi_L = 0.01$, respectively.

Finally, Figs. 5 and 6 demonstrate the true trajectories of the system coordinates under some initial condition and an admissible exogenous disturbance (the solid lines) and their estimates provided by the constructed pair (K^*, L^*) (the dotted lines).

7. CONCLUSIONS

In this paper, we have proposed a novel approach to rejecting nonrandom bounded exogenous disturbances in linear discrete-time control systems by means of dynamic output feedback. The approach is based on reducing the original problem to a matrix optimization problem, further solved by the gradient descent method. Of course, it is possible to use appreciably faster first-order minimization methods (in particular, the conjugate gradient method). Detailed verification of more efficient methods will be the subject of future research; for now, it is important to check the fundamental feasibility and effectiveness of the novel approach. Its applications-oriented feature is the possibility of limiting the magnitude of the observer and gain matrices.

Further studies may aim at generalizing the proposed approach to various robust problem statements.

APPENDIX

Proof of Lemma 2. Differentiation with respect to α is performed according to the results of Section 4.

To differentiate the objective function (9) with respect to K under the discrete Lyapunov equation constraint

$$\frac{1}{\alpha} A_{K,L} P A_{K,L}^{\mathrm{T}} - P + \frac{\delta^2}{1-\alpha} D_L D_L^{\mathrm{T}} = 0 \tag{A.1}$$

for the matrix P of the invariant ellipsoid, we assign an increment ΔK to K and denote the corresponding increment of P by ΔP:

$$\frac{1}{\alpha}(\mathcal{A} + M_1(K+\Delta K)N_1 + M_2 L N_2)(P+\Delta P)(\mathcal{A} + M_1(K+\Delta K)N_1 + M_2 L N_2)^{\mathrm{T}}$$
$$-(P+\Delta P) + \frac{\delta^2}{1-\alpha} D_L D_L^{\mathrm{T}} = 0.$$

With the notation ΔP kept for the principal part of the increment, we have

$$\frac{1}{\alpha}(A_{K,L}PA_{K,L}^{\mathrm{T}} + A_{K,L}P(M_1\Delta K N_1)^{\mathrm{T}} + M_1\Delta K N_1 P A_{K,L}^{\mathrm{T}} + A_{K,L}\Delta P A_{K,L}^{\mathrm{T}})$$
$$- (P + \Delta P) + \frac{\delta^2}{1-\alpha} D_L D_L^{\mathrm{T}} = 0.$$

Subtracting equation (A.1) from this expression yields

$$\frac{1}{\alpha} A_{K,L}\Delta P A_{K,L}^{\mathrm{T}} - \Delta P + \frac{1}{\alpha}(A_{K,L}P(M_1\Delta K N_1)^{\mathrm{T}} + M_1\Delta K N_1 P A_{K,L}^{\mathrm{T}}) = 0. \tag{A.2}$$

Let us calculate the increment of the function $f(K, L, \alpha)$ with respect to K by linearizing the corresponding values:

$$\Delta_K f(K, L, \alpha) = f(K + \Delta K, L, \alpha) - f(K, L, \alpha)$$
$$= \operatorname{tr} \mathcal{C}(P + \Delta P)\mathcal{C}^{\mathrm{T}} + \chi_K \|K + \Delta K\|_F^2 - (\operatorname{tr} \mathcal{C}P\mathcal{C}^{\mathrm{T}} + \chi_K \|K\|_F^2)$$
$$= \operatorname{tr} \mathcal{C}\Delta P\mathcal{C}^{\mathrm{T}} + \chi_K(\langle K + \Delta K, K + \Delta K\rangle - \langle K, K\rangle)$$
$$= \operatorname{tr} \mathcal{C}\Delta P\mathcal{C}^{\mathrm{T}} + \chi_K(\operatorname{tr} K^{\mathrm{T}}\Delta K + \operatorname{tr}(\Delta K)^{\mathrm{T}}K) = \operatorname{tr} \Delta P\mathcal{C}^{\mathrm{T}}\mathcal{C} + 2\chi_K \operatorname{tr} K^{\mathrm{T}}\Delta K.$$

From the dual Lyapunov equations (A.2) and (14) it follows that (see [11, Lemma A.1.1])

$$\operatorname{tr} \Delta P\mathcal{C}^{\mathrm{T}}\mathcal{C} = \frac{2}{\alpha} \operatorname{tr} Y M_1 \Delta K N_1 P A_{K,L}^{\mathrm{T}}$$

and consequently,

$$\Delta_K f(K, L, \alpha) = \frac{2}{\alpha} \operatorname{tr} Y M_1 \Delta K N_1 P A_{K,L}^{\mathrm{T}} + 2\chi_K \operatorname{tr} K^{\mathrm{T}}\Delta K$$
$$= 2\left\langle \frac{1}{\alpha} M_1^{\mathrm{T}} Y A_{K,L} P N_1^{\mathrm{T}} + \chi_K K, \Delta K \right\rangle.$$

Thus, we have established the relation (12).

Let us proceed to the differentiation of the function $f(K, L, \alpha)$ with respect to L: we assign an increment ΔL to L and denote the corresponding increment of P by ΔP:

$$\frac{1}{\alpha}(\mathcal{A} + M_1 K N_1 + M_2(L + \Delta L)N_2)(P + \Delta P)(\mathcal{A} + M_1 K N_1 + M_2(L + \Delta L)N_2)^{\mathrm{T}}$$
$$-(P + \Delta P) + \frac{\delta^2}{1-\alpha}\begin{pmatrix} D \\ D - (L + \Delta L)D_1 \end{pmatrix}\begin{pmatrix} D \\ D - (L + \Delta L)D_1 \end{pmatrix}^{\mathrm{T}} = 0.$$

With the notation ΔP kept for the principal part of the increment, we have

$$\frac{1}{\alpha}(A_{K,L}PA_{K,L}^{\mathrm{T}} + A_{K,L}P(M_2\Delta L N_2)^{\mathrm{T}} + M_2\Delta L N_2 P A_{K,L}^{\mathrm{T}})$$
$$-(P + \Delta P) + \frac{\delta^2}{1-\alpha}\left[D_L D_L^{\mathrm{T}} - \begin{pmatrix} 0 \\ \Delta L D_1 \end{pmatrix} D_L^{\mathrm{T}} - D_L \begin{pmatrix} 0 \\ \Delta L D_1 \end{pmatrix}^{\mathrm{T}}\right] = 0.$$

Subtracting equation (A.1) from this expression yields

$$\frac{1}{\alpha} A_{K,L}\Delta P A_{K,L}^{\mathrm{T}} - \Delta P + \frac{1}{\alpha}(A_{K,L}P(M_2\Delta L N_2)^{\mathrm{T}} + M_2\Delta L N_2 P A_{K,L}^{\mathrm{T}})$$
$$-\frac{\delta^2}{1-\alpha}\left[\begin{pmatrix} 0 \\ \Delta L D_1 \end{pmatrix} D_L^{\mathrm{T}} + D_L \begin{pmatrix} 0 \\ \Delta L D_1 \end{pmatrix}^{\mathrm{T}}\right] = 0. \tag{A.3}$$

We calculate the increment of the function $f(K, L, \alpha)$ with respect to L by linearizing the corresponding values:

$$\begin{gathered}\Delta_L f(K, L, \alpha) = f(K, L + \Delta L, \alpha) - f(K, L, \alpha) \\ = \operatorname{tr} \mathcal{C}(P + \Delta P)\mathcal{C}^{\mathrm{T}} + \chi_L \|L + \Delta L\|_F^2 - (\operatorname{tr} \mathcal{C} P \mathcal{C}^{\mathrm{T}} + \chi_L \|L\|_F^2) \\ = \operatorname{tr} \mathcal{C} \Delta P \mathcal{C}^{\mathrm{T}} + \chi_L(\langle L + \Delta L, L + \Delta L\rangle - \langle L, L\rangle) \\ = \operatorname{tr} \mathcal{C} \Delta P \mathcal{C}^{\mathrm{T}} + \chi_L(\operatorname{tr} L^{\mathrm{T}} \Delta L + \operatorname{tr} (\Delta L)^{\mathrm{T}} L) = \operatorname{tr} \Delta P \mathcal{C}^{\mathrm{T}} \mathcal{C} + 2\chi_L \operatorname{tr} L^{\mathrm{T}} \Delta L.\end{gathered}$$

From the dual Lyapunov equations (A.3) and (14) it follows that

$$\operatorname{tr} \Delta P \mathcal{C}^{\mathrm{T}} \mathcal{C} = 2\operatorname{tr} Y \left[\frac{1}{\alpha} M_2 \Delta L N_2 P A_{K,L}^{\mathrm{T}} - \frac{\delta^2}{1-\alpha} \begin{pmatrix} 0 \\ \Delta L D_1 \end{pmatrix} D_L^{\mathrm{T}}\right]$$

and consequently,

$$\begin{gathered}\Delta_L f(K, L, \alpha) \\ = 2\operatorname{tr} Y \left[\frac{1}{\alpha} M_2 \Delta L N_2 P A_{K,L}^{\mathrm{T}} - \frac{\delta^2}{1-\alpha} \begin{pmatrix} 0 \\ \Delta L D_1 \end{pmatrix} D_L^{\mathrm{T}}\right] + 2\chi_L \operatorname{tr} L^{\mathrm{T}} \Delta L \\ = 2\operatorname{tr} \left[\frac{1}{\alpha} N_2 P A_{K,L}^{\mathrm{T}} Y M_2 \Delta L - \frac{\delta^2}{1-\alpha} D_1 D_L^{\mathrm{T}} Y \begin{pmatrix} 0 \\ I \end{pmatrix} \Delta L\right] + 2\chi_L \operatorname{tr} L^{\mathrm{T}} \Delta L \\ = 2\left\langle \frac{1}{\alpha} M_2^{\mathrm{T}} Y A_{K,L} P N_2^{\mathrm{T}} - \frac{\delta^2}{1-\alpha} \begin{pmatrix} 0 & I \end{pmatrix} Y D_L D_1^{\mathrm{T}} + \chi_L L, \Delta L \right\rangle,\end{gathered}$$

which finally gives formula (13).

The proof of Lemma 2 is complete.

Proof of Lemma 3. Consider a sequence of stabilizing matrix pairs $\{K_j, L_j\} \in \mathcal{S}$ such that

$$(K_j, L_j) \to (K, L) \in \partial \mathcal{S},$$

i.e., $\rho(A_{K,L}) = 1$. In other words, for any $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that

$$|\rho(A_{K_j,L_j}) - \rho(A_{K,L})| = 1 - \rho(A_{K_j,L_j}) < \epsilon \quad \text{for all } j \geqslant N(\epsilon).$$

Let P_j be the solution of the discrete Lyapunov equation (10) associated with the pair (K_j, L_j), i.e.,

$$\frac{1}{\alpha_j} A_{K_j,L_j} P_j A_{K_j,L_j}^{\mathrm{T}} - P_j + \frac{\delta^2}{1-\alpha_j} D_{L_j} D_{L_j}^{\mathrm{T}} = 0,$$

and Y_j be the solution of its dual Lyapunov equation

$$\frac{1}{\alpha_j} A_{K_j,L_j}^{\mathrm{T}} Y_j A_{K_j,L_j} - Y_j + \mathcal{C}^{\mathrm{T}} \mathcal{C} + \varepsilon I = 0.$$

By [12, Lemma A.1], we have

$$\begin{gathered}f(K_j, L_j) = \operatorname{tr} P_j(\mathcal{C}^{\mathrm{T}} \mathcal{C} + \varepsilon I) + \chi_K \|K_j\|_F^2 + \chi_L \|L_j\|_F^2 \geqslant \operatorname{tr} P_j(\mathcal{C}^{\mathrm{T}} \mathcal{C} + \varepsilon I) \\ = \operatorname{tr} Y_j \frac{\delta^2}{1-\alpha_j} D_{L_j} D_{L_j}^{\mathrm{T}} \geqslant \frac{\delta^2}{1-\alpha_j} \lambda_{\min}(Y_j) \operatorname{tr} (D_{L_j} D_{L_j}^{\mathrm{T}}) \geqslant \frac{\delta^2}{1-\alpha_j} \frac{\lambda_{\min}(\mathcal{C}^{\mathrm{T}} \mathcal{C} + \varepsilon I)}{1 - \sigma_{\min}^2(A_{K_j,L_j})} \operatorname{tr} (D_{L_j} D_{L_j}^{\mathrm{T}}) \\ \geqslant \frac{\delta^2}{1 - \rho^2(A_{K_j,L_j})} \frac{\varepsilon}{1 - \sigma_{\min}^2(A_{K_j,L_j})} \|D_{L_j}\|_F^2 \geqslant \frac{\delta^2}{\epsilon} \frac{\varepsilon}{1 - \sigma_{\min}^2(A_{K_j,L_j})} \|D_{L_j}\|_F^2 \xrightarrow[\epsilon \to 0]{} +\infty\end{gathered}$$

since

$$\rho^2(A_{K_j,L_j}) < \alpha_j < 1.$$

On the other hand,

$$f(K_j, L_j) = \operatorname{tr} P_j(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I) + \chi_K\|K_j\|_F^2 + \chi_L\|L_j\|_F^2$$
$$\geqslant \chi_K\|K_j\|_F^2 \geqslant \chi_K\|K_j\|^2 \xrightarrow[\|K_j\|\to+\infty]{} +\infty$$

and

$$f(K_j, L_j) = \operatorname{tr} P_j(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I) + \chi_K\|K_j\|_F^2 + \chi_L\|L_j\|_F^2$$
$$\geqslant \chi_L\|L_j\|_F^2 \geqslant \chi_L\|L_j\|^2 \xrightarrow[\|L_j\|\to+\infty]{} +\infty.$$

The proof of Lemma 3 is complete.

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